# State variable changes to avoid non computational issues

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## 1 Abstract

This paper is about the numerical simulation of nonlinear analog circuits with ”switch” components, such as diodes. A ”switch” component is an electrical device that may or may not conduct, depending on the state of the circuit. The problem with ”switch” components is that the topology of the circuit is variable and so, apparently, it is not possible to describe the system with a single differential equation and solve it using standard numerical methods. This paper shows how to choose an appropriate state variable and overcome the above difficulties.

## 2 A test example

Let’s consider the following circuit.

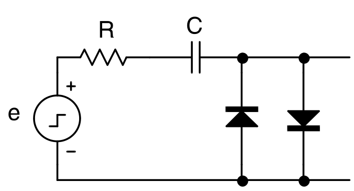
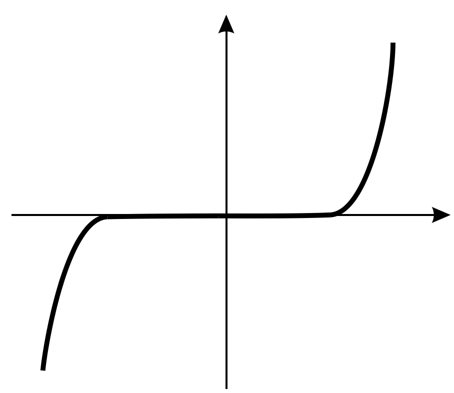



*Figure 1: Example circuit*

If the voltage on the diodes is below their threshold, no current is flowing through the diodes and so no current is flowing into the resistor and the capacitor; the circuit is open. When the voltage gets higher than the diode threshold, the circuit becomes a standard RC filter. We have just seen two different topologies that the same circuit can have, depending on its state.

The couple of diodes has a memory-less nonlinear transfer function, like the one in figure 1. Note that the current circuit is only an example and so, instead of diodes, we could have other switch components with a different transfer function, like the grid of a triode. But the ideas we are going to expose will remain the same.



$i_d = f(v_d)$

*Figure 2: Nonlinear transfer function*

Let’s call $i_d$ the current flowing into the diodes and $v_d$ the voltage on the diode. A standard way to proceed into the analysis of the circuit, is to take the current $i$ will be as the state variable of the system. The equations of the system will become:

$$\begin{cases} e - iR - v_c - v_d = 0 \\ v_c(t) = \frac{1}{C}\int_0^t i(s)ds \\ i(t) = f(v_d(t)) \end{cases}$$

Where $v_c$ is the voltage on the capacitor and $v_d$ is the voltage on the diodes. Remembering that $i$ is the state variable, in order to be integrated, the above system should be written as:

$$e - iR - \frac{1}{C}\int_0^t i(s)ds - f^{-1}(i) = 0$$

But this is not possible because $i_d = f(v_d)$ is not invertible.

But actually, this is not a real issue. All we have to do is choosing another state variable. Taking $v_d$, the system can be written as:

$$e - f(v_d)R - \frac{1}{C}\int_0^t f(v_d(s))ds - v_d = 0 \qquad (1)$$

Now the equation has a valid analytic form.

# 3 Analysis of the solution

The aim of this section is to give an example of how we can study if the solution of (1) exist and is unique. First of all (1) should be rewritten in differential form:

$$v_d' = \frac{e' - \frac{1}{C}f(v_d)}{1 + f'(v_d)R}$$

Then we have to require that the second term of this equation is continuous and satisfies the Lipschitz condition. To have continuity, the denominator should satisfy

$$|1 + f'(v_d)R| > \epsilon$$

For "clipping" transfer functions like the one of diodes, it is $f'(v_d) \geq 0$ and so $1 + f'(v_d)R \geq 1$.

Instead of Lipschitz condition, we could ask that

$$\frac{e' - \frac{1}{C}f(v_d)}{1 + f'(v_d)R} \in C^1$$

This is a stronger condition; if it is satisfied, then also Lipschitz condition is satisfied. It is easy to see that in clipping devices, with $f'(v_d) \geq 0$, the above condition is true if and only if $f \in C^1$.

Considering that in real applications $f$ is not defined analytically, but is often a regular function interpolating some measurement point, asking $f \in C^1$ does not limit the validity of this technique.

# 4 The numerical discretization

Now we are ready to study a numerical technique to solve (1). Let's consider a discretization step of $h$, such that

$$t_k = kh$$

We want to find a sequence $\{y_k\}$ that approximates the real value of $v_d$ in the grid points:

$$y_k \simeq v_d(t_k)$$

Starting from equation (1), we have to give a numerical approximation of the integral. A common choice in the audio applications is the trapezoidal rule, that corresponds to the bilinear transform method. One of its main advantages is that it is a one step method but it has quadratic order of convergence. The discrete equation is:

$$e_k - e_{k-1} - y_k + y_{k-1} = (\frac{h}{2C} + R)f(y_k) + (\frac{h}{2C} - R)f(y_{k-1})$$

And, after rearranging the terms, the equation becomes:

$$-Af(y_k) - K = y_k \qquad (2)$$

Where

$$A = \frac{h}{2C} + R$$

$$K = e_{k-1} - e_k + (\frac{h}{2C} - R)f(y_{k-1}) - y_{k-1}$$

We can easily see that there exist one and only one solution of (2). Rewriting it as

$$f(y_k) = -A^{-1}(y_k + K)$$

we note that the solution is the intersection of the graph of $y = f(x)$ and the line $y = -A^{-1}(x + K)$, as shown in the picture below.

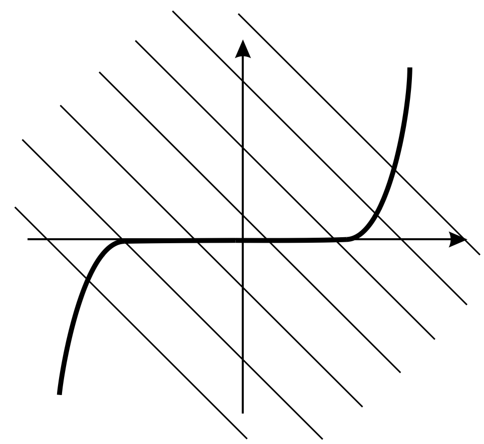


*Figure 3*

Because the lines have a negative slew and the nonlinear transfer function is monotone with a positive slew, there is only one point of intersection, and so the solution is unique.

A solution of (2) can be easily found with standard numerical methods, like Newton-Raphson or bisection algorithm, which globally converge thanks to the regularity and monotonicity of the function. A little trick to improve the speed of convergence is to use a "hot start" at each iteration; this consists in initializing the iterative methods using the solution at the previous discretization point, which is a first order approximation of the new solution.

# 5 Comparison with a simplified method

In the previous sections we studied a very simple example like a model; but in spite of its simplicity, the analysis and the computational models are not elementary. Generally, in practical applications like the signal processing in the musical field, simplified models are used instead of the one presented here. Giving up the possibility to obtain an exact simulation of the circuit, they are computationally cheaper and easier to implement. For example, the circuit considered in the previous sections can be approximated by a one pole high pass filter (with cutoff frequency at $\frac{1}{2\pi RC}$) followed by a nonlinear waveshaper with the transfer function of the diodes. This solution is commonly used into the DSP models in tube preamplifier simulators. So we may ask if such an approximation can give sufficiently good accuracy.

We are going to present the results of a numerical experiment that can be quite illuminating. The experiment is based on the same circuit of the previous sections, taking $R = 10K\Omega$, $C = 22nF$ and the generator $e$ as a sine source with a frequency of 60Hz. For this circuit both the exact and the simplified models have been built. In the first simulation, the amplitude of the source signal was 20V peak to peak, so the signal was heavily clipped by the diodes. In the second run, the same circuit has been simulated using an input voltage of 0.5V peak to peak. Here are the graphics with the results:

In the first case (*figure 4*) the source signal is heavily clipped; this means that the diodes are conducing in about all the period of the signal, current is flowing through the capacitor and so the RC network is behaving like the HPF of the approximated model. But in the second case (*figure 5*), diodes are not conducing, so in the real case no current is flowing through the RC network and no filtering effect is performed. This is a big difference between the approximated model.

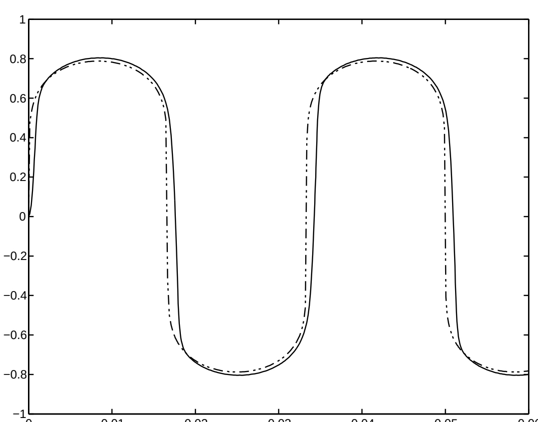



*Figure 4: Simulation of the circuit with an input voltage of 20V peak to peak. The solid line is the exact simulation. The dotted line is the simulation with the approximated model.*

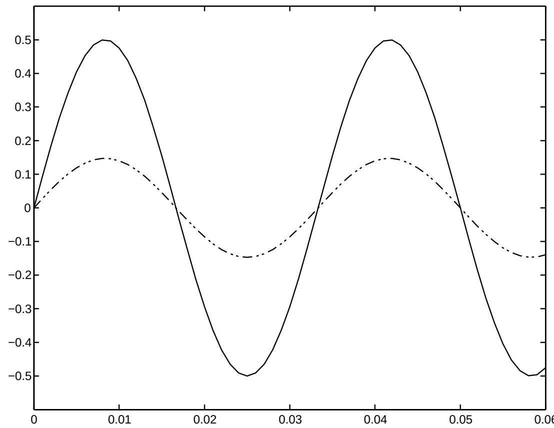



*Figure 5: Simulation of the circuit with an input voltage of 0.5V peak to peak. The solid line is the exact simulation. The dotted line is the simulation with the approximated model.*

# 6 Conclusions

This paper would like to introduce, from an operative point of view, some aspects of the simulation of circuits with ”switching” components. It would not to be a complete theoretic treatment, but it would like to give a complete example of how to build models when switching components are present, how to analyze them and give a numerical method for the simulation. This treatment can be applied without relevant changes to all the saturating components, like transistor, tubes and saturating magnetic cores, which are almost all the nonlinearities found in audio applications.

# 7 Acknowledgment

we would like to thank Gianpaolo Borin who inspired this work and Pierre Richemond for his revision.

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